1.8 Vector-Valued Functions

Tuesday, February 2, 2021 12:17 PM

<u>Naive definition</u>: Vector-valued fcn is a fcn from $\mathbb{R}$ to $\mathbb{R}^3$

i.e. $\vec{f}$ sends $t \in \mathbb{R}$ to $\vec{f}(t) \in \mathbb{R}^3$

<u>But</u> what about:

$\vec{f}(t) = (t, 3t, 1/t)$

→ not defined at $t = \emptyset$

so it's a fcn from $\mathbb{R} \setminus \{0\}$ = set of nonzero real numbers to $\mathbb{R}^3$

<u>Better definition</u>: A vector-valued fcn is a fcn from a subset D of $\mathbb{R}$ to $\mathbb{R}^3$.

e.g. $\vec{f}(t) = \left(\frac{1}{1-t}, \sqrt{t}, \sin(t)\right)$

is defined for $t \geq 0$ and $t \neq 1$.

i.e. $D = [0, 1) \cup (1, \infty)$

= set of real numbers that are neither negative <u>nor</u> equal to 1.

Can think of as a parametric eq. in $\mathbb{R}^3$

e.g.

<u>line</u>: $\vec{f}(t) = \vec{x_0} + t\vec{v}$

<u>helix</u>: $\vec{f}(t) = (\cos t, \sin t, t)$

Can write vector-val'd fcn as:

① $\vec{f}(t) = f_1(t)\vec{i} + f_2(t)\vec{j} + f_3(t)\vec{k}$

② $\vec{f}(t) = (f_1(t), f_2(t), f_3(t))$

<u>Note</u>: A lot of vector valued calc is just a matter of doing single var calc in each coord separately (true of limits, continuity, derivative)

→ Becomes something new when we do dot & cross products

Limits

Definition: If $\vec{F}$ is a v-v f on D and $a \in D$ and $\vec{e} \in \mathbb{R}^3$, we say:

$$\lim_{t \to a} \vec{F}(t) = \vec{e}$$

if one of 2 eq. conditions holds:

(A) $\forall \varepsilon > 0$, $\exists \delta > 0$
s.t. if $|t-a| < \delta$
then distance $(\vec{F}(t), \vec{e}) < \varepsilon$

$$\text{distance}(\vec{F}(t), \vec{e}) = ||\vec{e} - \vec{F}(t)||$$

(B) for $i = 1,2,3$ we have
$$\lim_{t \to a} f_i(t) = e_i$$

Why are (A) & (B) equivalent?

- The ith coord of $\vec{e} - \vec{F}(t)$ is $e_i - f_i(t)$.
- Def (A) says that we can make $||\vec{e} - \vec{F}(t)||$ small when t is close to a.
  Def (B) says that we can make $|e_i - f_i(t)|$ small when t is close to a.
- They are equivalent bc a vector is small in magnitude iff its components are all small in absolute value.
- Qualitatively:
  For a vector $\vec{v} = (v_1, v_2, v_3)$ each of $|v_1|, |v_2|$, and $|v_3|$ is $\leq ||\vec{v}||$
  and
  $||\vec{v}|| \leq |v_1| + |v_2| + |v_3|$

Continuity

Suppose $\vec{f}(t)$ is defined as a v-v f for $t, a \in D$. Then we say $\vec{f}$ is continuous if either of the two eq. conds hold:

(A) $\lim_{t \to a} f(t) = \vec{f}(a)$

(B) each of $f_1(t), f_2(t), f_3(t)$ is cont at a.

Derivatives

We define (for $a \in D$):

$$f'(a) = \lim_{h \to 0} \frac{\vec{f}(a+h) - \vec{f}(a)}{h}$$

$$= \lim_{t \to a} \frac{\vec{f}(t) - \vec{f}(a)}{t-a}$$

We say $\vec{f}$ is differentiable at a if this limit exists.

equivalently

$\vec{f}$ is differentiable iff f_1, f_2, and f_3 are differentiable

P iff Q means if P then Q & if Q then P.

If $\vec{f}$ is differentiable at a, then
$\vec{f}'(a) = (f_1'(a), f_2'(a), f_3'(a))$

New idea: Derivative is a vector not a scalar.
ie, has magnitude & direction

Physical Interpretation

For an obj whose position at time t is given by $\vec{f}(t)$, its velocity is $\vec{f}'(t)$, its speed is $\|\vec{f}(t)\|$, the direction of $f'(t)$ is the direction the obj is moving.

acceleration is:

$$\vec{f}''(t) = \frac{dF}{dt} f'(t) \leftarrow \text{Check this}$$

$\vec{F}''(t) = \frac{dF}{dt} f'(t) \leftarrow$ Check this

In physics

Basic Properties of Deriv's

Same as in single var

① $\vec{F}'(t) = \emptyset$ iff $\vec{F}$ is a constant fcn (on each interval)

- In general, for any D, if $\vec{F}$ is const, then $\vec{F}'(t) = 0$
- If D is an interval like (a,b) or $[a,b]$ or half-open, then if $\vec{F}'(t) = 0$ then $\vec{F}(t)$ is constant.

② Linearity

If $m, n \in \mathbb{R}$ and $\vec{f}$ and $\vec{g}$ are diff'able v-v f, then

$$\frac{d}{dt}(m\vec{f}(t) + n\vec{g}(t)) = m f'(t) + n g'(t)$$

derivative of a linear combination is a linear combination of the derivatives

Different in MVC

① New kinds of products:

- multiply vector by scalar
  output: vector
- dot product of 2 vectors
  output: scalar
- cross product of 2 vectors
  output: vector

$\Rightarrow$ 3 product rules for derivatives

Let $\vec{f}, \vec{g}$ be v-v f on $D \subseteq \mathbb{R}$

and $u(t)$ be a scalar-valued func on D. Then:

Vector
scalar

① $\frac{d}{dt}(u(t)\vec{F}(t))$

$= u'(t) F(t) + u(t)\vec{F}'(t)$

② $\frac{d}{dt}(\vec{F}(t)\cdot\vec{g}(t))$

$= \vec{F}'(t)\cdot\vec{g}(t) + \vec{F}(t)\cdot\vec{g}'(t)$

ie we wrote a single-var calc deriv in terms of dot prods of vector derivatives

③ $\frac{d}{dt}(\vec{F}(t)\times\vec{g}(t))$

$= \vec{F}'(t)\times\vec{g}(t) + \vec{F}(t)\times\vec{g}'(t)$

⚠ order of cross product matters

Let's use dot product for some vector calculus geometry

Consider speed $\|\vec{F}'(t)\|$

Actually, let's consider speed^2

$= \|\vec{F}'(t)\|^2 = \vec{F}'(t)\cdot F'(t)$

Two ways:

Ⓐ Use ①

$\frac{d}{dt}(\text{speed}^2) = \frac{d}{dt}(\vec{F}'\cdot\vec{F}')$

$= \left(\frac{d}{dt}\vec{F}'\right)\cdot\vec{F}' + \vec{F}'\cdot\frac{d}{dt}(\vec{F}')$

$= \vec{F}''\cdot\vec{F}' + \vec{F}'\cdot\vec{F}''$

$= 2F'\cdot\vec{F}''$

Ⓑ $\frac{d}{dt}(\text{speed}^2)$

single-var product rule)

(B) $\frac{d}{dt}(\text{speed}^2)$ single-var product rule

$= 2(\text{speed}) \cdot \frac{d}{dt}(\text{speed})$

$= 2\|\vec{F}'\| \cdot \frac{d\|\vec{F}\|}{dt}$

$= \frac{d}{dt}\|\vec{F}\|^2$

Conclusion

1. When is speed constant?

Note: speed is const iff speed^2 is const.

and this is the iff

$\frac{d}{dt}\text{speed}^2 = \varnothing = 2\vec{F}' \cdot \vec{F}''$

Q: When is $\vec{F}' \cdot \vec{F}'' = 0$?

A: When $\vec{F}' \perp \vec{F}''$

so speed doesn't change (ie, only the direction changes) precisely when (iff) the acceleration is perpendicular to the direction of motion

2. Formula for $\frac{d}{dt}(\text{speed})$. How?

Note: set (A) equal to (B)

$2(\text{speed})\frac{d}{dt}(\text{speed}) = \frac{d}{dt}(\text{speed}^2)$

$= 2\vec{F}' \cdot \vec{F}''$

$\Rightarrow (\text{speed}) \cdot \frac{d}{dt}(\text{speed}) = 2\vec{F}' \cdot \vec{F}''$

$\Rightarrow \frac{d(\text{speed})}{dt} = \frac{d\|\vec{F}'(t)\|}{dt} = \frac{2\vec{F}' \cdot \vec{F}''}{\|F'\|}$

<u>See in book</u> similar reasoning with $\vec{F}$ in place of $\vec{F}'$ shows that

① $\|\vec{F}\|$ is const, ie $\vec{F}(t)$ is contained in a circle, iff $\vec{F} \perp \vec{F}'$.

② $\dfrac{d\|\vec{F}(t)\|}{dt} = \dfrac{d\rho}{dt} = \dfrac{\vec{F} \cdot \vec{F}'}{\|\vec{F}\|}$